Analysis 1A — Tutorial 3

Christian Jones: University of Bath

October 2022

Table of Contents

# Introduction

Here is a version of Tutorial Question 3 off of Problem Sheet 3 with an alternative solution for part c). Parts a) and b) are included for completeness.

Example 1 (PS3 Question 3)

1. Show that
2. and that equality holds only if .
3. Show that
4. Prove that

## Part a)

Solution.

We have that for any

from which rearranging gives

Now,

So equality holds *if and only if*

## Part b)

Solution.

Consider the second inequality first. Note that since , , and so

Square rooting this result gives us that

Next, we have that

Again, square rooting gives us that

## Part c)

Solution.

Firstly for ,

For , we have

$$\begin{align}
\lvert \sqrt{1 + x^2} - \sqrt{1 + y^2}\rvert &= \frac{1 + \lvert x^2 - (1 + y^2) \rvert}{\sqrt{1 + x^2} + \sqrt{1 + y^2}},\tag{\*}\\
&= \frac{\lvert x^2 - y^2 \rvert}{\sqrt{1 + x^2} + \sqrt{1 + y^2}},\notag\\
&= \frac{\lvert x + y \rvert \lvert x - y \rvert}{\sqrt{1 + x^2} + \sqrt{1 + y^2}}.\notag
\end{align}$$

Now,

(This can be seen by squaring both sides of each inequality)

So,

Therefore,

$$\begin{align}
\lvert \sqrt{1 + x^2} - \sqrt{1 + y^2}\rvert &= \frac{\lvert x + y \rvert \lvert x - y \rvert}{\sqrt{1 + x^2} + \sqrt{1 + y^2}},\notag\\
&\leq \frac{\lvert x - y \rvert\lvert x + y \rvert}{\lvert x + y \rvert},\tag{\*\*}\\
& = \lvert x - y \rvert,\notag
\end{align}$$

as required!

You might have a few questions about this:

**Q1)** Why is 3c) done in this way?

A1) It’s an alternative way to the one in the model solutions, but I think it’s good because it uses some techniques that are useful for the sequences part of the course (e.g. the triangle inequality and step (\*)).

**Q2)** Where on Earth did the case come from?

A2) If you look at (\*\*), this expression doesn’t work if , so you need to consider this separately. It’s not an obvious case until you actually reach (\*\*), but once you realise it, it’s an easy thing to add to the start of your solution.

**Q3)** What about (\*)? Where does this come from?

A3) Recall for ,

Taking we have that